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Modulated waves and helicoidal pseudospherical surfaces in nonlinear inhomogeneous elasticity

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Abstract
The nonlinear wave equation

$$\frac{\partial^2 T}{\partial X^2} = \frac{\partial}{\partial t}\left[\frac{\partial T}{\partial t}/(1 + X^2 + T^2)^2\right]$$

provides a Lagrangian description of one-dimensional stress propagation in a class of model inhomogeneous ideally hard elastic materials. It is associated with the geometry of pseudospherical surfaces and, as such, is integrable and admits an auto-Bäcklund transformation. Here, modulated waves associated with helicoidal pseudospherical surfaces are investigated.

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1. Introduction

In [1], a novel avatar of the classical sine-Gordon equation was introduced which describes one-dimensional stress propagation in a class of ideally hard inhomogeneous elastic materials which exhibit locking strain. The classical Bäcklund transformation for the construction of pseudospherical surfaces was used to induce an auto-Bäcklund transformation for the stress propagation equation. Subsequently, in [2], the geometric connection with pseudospherical surfaces was exploited to construct exact solutions of this nonlinear wave equation which correspond, in turn, to the Beltrami pseudosphere and to a 'two-soliton' pseudospherical surface obtained via a double application of the classical Bäcklund transformation. However, in the physical (X, t)-space, even in the basic Beltrami pseudosphere case corresponding to a single-soliton solution of the sine-Gordon equation, the associated exact solution of the stress propagation equation has complex behaviour with the appearance of shock fronts for certain parameter ranges. Here, a direct approach involving a base-modulated wave ansatz for the stress distribution is adopted. The associated class of exact solutions turns out to correspond to helicoidal pseudospherical surfaces and, in general, involves elliptic integrals. A phase plane analysis is undertaken which reveals intriguing features in the 'small' and 'large' amplitude limits.

2. A class of ideally hard elastic materials

Here, we review the derivation of the model laws introduced in [1] which are descriptive of a class of ideally hard elastic inhomogeneous materials. Thus, the Lagrangian equations of (1+1)-dimensional elasticity are

$$\epsilon_t = v_X, \qquad \rho_0 v_t = T_X, \tag{2.1}$$

together with an appropriate constitutive law, here taken to adopt the form

$$T = T(\epsilon, X). \tag{2.2}$$

In the above, T denotes the stress, $\epsilon = \rho_0/\rho - 1$ is the stretch, v is the material velocity, while ρ and ρ_0 designate the density of the elastic medium in its deformed and undeformed states respectively. X denotes the Lagrangian spatial coordinate and t time. Hereafter, we set $\rho_0 = 1$. If the constitutive law (2.2) adopts the form

$$\epsilon = \epsilon(T, X), \tag{2.3}$$

then elimination of the material velocity between the constituent Lagrangian equations of (2.1) leads to the nonlinear wave equation

$$T_{XX} = (\epsilon_T T_t)_t, \tag{2.4}$$

where $\epsilon_T = \epsilon_T|_X$.

In order to establish a geometric connection, it is observed that the stress propagation equation (2.4) admits the two elementary conservation laws

$$y_t = T_X, \qquad y_X = \epsilon_T T_t \tag{2.5}$$

and

$$z_t = -T + XT_X, \qquad z_X = \epsilon_T X T_t. \tag{2.6}$$

Any solution $T = T(X, t)$ of the stress propagation equation uniquely determines a surface Σ in $\mathbb{R}^3$ with position vector

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad x = t. \tag{2.7}$$

Relations (2.5) and (2.6) may be compactly encapsulated, with (2.4) as their compatibility condition, in terms of differential forms, namely

$$\mathrm{d}z = -T\,\mathrm{d}x + X\,\mathrm{d}y, \qquad \epsilon_T\,\mathrm{d}T \wedge \mathrm{d}X = \mathrm{d}x \wedge \mathrm{d}y \tag{2.8}$$

(and, implicitly, $\mathrm{d}T \wedge \mathrm{d}x = \mathrm{d}X \wedge \mathrm{d}y$). Alternatively, if x and y are taken as new independent variables, then $(2.8)_1$ yields

$$T = -z_x, \qquad X = z_y, \tag{2.9}$$

while $(2.8)_2$ becomes

$$z_{xx}z_{yy} - z_{xy}^2 = -\frac{1}{\epsilon_T(-z_x, z_y)}. \tag{2.10}$$

Accordingly, the Gaussian curvature [3]

$$\mathcal{K} = \frac{z_{xx}z_{yy} - z_{xy}^2}{\left(1 + z_x^2 + z_y^2\right)^2} \tag{2.11}$$

of the surface Σ is given by

$$\mathcal{K} = -\frac{1}{\left(1 + z_x^2 + z_y^2\right)^2 \epsilon_T(-z_x, z_y)}. \tag{2.12}$$

The above analysis shows that, in geometric terms, the prescription of the constitutive laws of the form (2.3) corresponds to a particular choice of the Gaussian curvature

$$\mathcal{K} = \mathcal{K}(z_x, z_y). \tag{2.13}$$

In the simplest case

$$\mathcal{K} = -1, \tag{2.14}$$

associated with a pseudospherical surface Σ, integration of

$$\epsilon_T = \frac{1}{(1 + X^2 + T^2)^2} \tag{2.15}$$

yields the constitutive laws in parametric form, namely

$$\begin{aligned} T &= \sqrt{1 + X^2}\tan\theta \\ \epsilon &= \frac{1}{2(1 + X^2)^{3/2}}(\theta + \sin\theta\cos\theta) + \alpha(X). \end{aligned} \tag{2.16}$$

Here, we proceed with the condition

$$T|_{\epsilon=0} = 0, \tag{2.17}$$

so that the function of integration $\alpha(X)$ vanishes. Accordingly, we obtain a stress–strain law $T = T(X, \epsilon)$ in the implicit form

$$\epsilon = \frac{1}{2(1 + X^2)^{3/2}}\left[\arctan\left(\frac{T}{\sqrt{1 + X^2}}\right) + \frac{T\sqrt{1 + X^2}}{1 + X^2 + T^2}\right]. \tag{2.18}$$

The model constitutive law (2.16) associated with pseudospherical surfaces is characterized by the signal speed

$$\mathcal{A} = \sqrt{T_\epsilon} = 1 + X^2 + T^2, \tag{2.19}$$

and is such that

$$T_\epsilon \to \infty \tag{2.20}$$

and

$$\epsilon \to \epsilon_L = \frac{\pi}{4(1 + X^2)^{3/2}} \tag{2.21}$$

as $T \to \infty$ for fixed X. Hence, the model law $T = T(\epsilon, X)$ given parametrically by (2.16) describes inhomogeneous ideally hard elastic materials with locking strain ϵ_L. It is remarked that ideally hard elastic behaviour is exhibited in the dynamic compression of such materials as dry sand, saturated soil and clay silt (q.v. [4, 5] and work cited therein).

Insertion of (2.15) into (2.4) now produces the nonlinear wave equation

$$T_{XX} = \left[\frac{T_t}{(1 + X^2 + T^2)^2}\right]_t \tag{2.22}$$

descriptive of (1+1)-dimensional stress propagation in the class of ideally hard elastic materials with constitutive law given by the parametric relations (2.16). A Bäcklund transformation which leaves (2.22) invariant is readily constructed via the classical result for pseudospherical surfaces [1, 6].

3. Modulated waves and helicoidal pseudospherical surfaces

The nonlinear stress propagation equation (2.22) is privileged in that it is but another avatar of the classical integrable sine-Gordon equation

$$\omega_{uv} = \sin\omega, \tag{3.1}$$

where u and v constitute asymptotic coordinates on the pseudospherical surface Σ. The classical Bäcklund transformation for pseudospherical surfaces may be employed to generate non-trivial solutions of (2.22) corresponding to solitons and breathers of the sine-Gordon equation (3.1). The action of the Bäcklund transformation on the nonlinear wave equation (2.22) has been analysed in [2]. Here, we adopt a direct approach which delivers modulated travelling wave solutions. These may then be used as seeds in the iterative application of the Bäcklund transformation. Thus, if we introduce the modulated travelling wave ansatz

$$T = \sqrt{1+X^2}\,\Phi(t - S(X)), \tag{3.2}$$

it is seen that

$$T = \sqrt{1+X^2}\,\Phi(s), \qquad s = t - A\arctan X, \tag{3.3}$$

and Φ is determined by the nonlinear ordinary differential equation

$$A^2\Phi'' + \Phi = \left[\frac{\Phi'}{(1+\Phi^2)^2}\right]' \tag{3.4}$$

or, equivalently,

$$\Phi'' = -\Phi\frac{(1+\Phi^2)^3 + 4\Phi'^2}{(1+\Phi^2)[A^2(1+\Phi^2)^2 - 1]}, \tag{3.5}$$

where A constitutes an arbitrary constant of integration.

In order to identify the class of pseudospherical surfaces represented by the present class of modulated waves, we first introduce the integro-differential from

$$\left[\frac{1}{(1+\Phi^2)^2} - A^2\right]\Phi' = I[\Phi], \qquad I[\Phi](s) = \int \Phi(s)\,\mathrm{d}s \tag{3.6}$$

of the second-order equation (3.4). It is then readily verified that the potentials y and z defined by relations (2.5) and (2.6) are given explicitly by

$$y = I[\Phi]\sin\rho - A\Phi\cos\rho, \qquad z = -I[\Phi]\cos\rho - A\Phi\sin\rho, \tag{3.7}$$

where

$$\sin\rho = \frac{X}{\sqrt{1+X^2}}, \qquad \cos\rho = \frac{1}{\sqrt{1+X^2}} \tag{3.8}$$

with irrelevant constants of integration being omitted. Thus, it is deduced that

$$r^2 := y^2 + z^2 = I^2[\Phi] + A^2\Phi^2 =: h(s) = h(t - A\rho), \tag{3.9}$$

and if $p(s)$ is introduced according to

$$A\tan p = \frac{I[\Phi]}{\Phi}, \tag{3.10}$$

then

$$\frac{z}{y} = \tan(p+\rho) =: \tan\phi. \tag{3.11}$$

Combination of relations (3.9) and (3.11) now shows that

$$x = t = k(r^2) + A\phi \tag{3.12}$$

for some function k depending on h and p. Hence, the position vector (2.7) adopts the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k(r^2) + A\phi \\ r\cos\phi \\ r\sin\phi \end{pmatrix}, \tag{3.13}$$

which corresponds to a class of helicoidal pseudospherical surfaces with pitch $2\pi A$ (see [3]). Indeed, the general class of helicoidal pseudospherical surfaces may be generated via the modulated wave ansatz (3.2). In terms of the classical sine-Gordon equation (3.1), these helicoidal surfaces correspond to the pendulum reduction,

$$c\omega'' = \sin\omega, \qquad \omega = \omega(u + cv), \qquad c \in \mathbb{R}, \tag{3.14}$$

with general solution expressible in terms of Jacobi elliptic functions.

4. Elliptic integral reduction. Phase plane analysis

The nonlinear Φ-equation (3.4) admits the first integral

$$\Phi'^2 = \frac{(1+\Phi^2)^3[-A^2(1+\Phi^2)^2 + B(1+\Phi^2) - 1]}{[A^2(1+\Phi^2)^2 - 1]^2}, \tag{4.1}$$

whence

$$\alpha' = \pm R(\alpha, \sqrt{P_4(\alpha)}), \qquad \alpha = \Phi^2, \tag{4.2}$$

where R is a rational function of its arguments and $P_4(\alpha)$ is a polynomial of degree 4. Accordingly,

$$s = t - A\arctan X = E(\Phi^2), \tag{4.3}$$

where E denotes the elliptic integral

$$E = \pm\int \frac{\mathrm{d}\alpha}{R(\alpha, \sqrt{P_4(\alpha)})}. \tag{4.4}$$

Thus, the general solution of (3.4) may in principle be expressed implicitly in terms of elliptic integrals of the first, second and third kind. However, in view of the opaque nature of such an implicit solution, it proves instructive to examine its behaviour in the phase plane. In this connection, it is noted that the second factor in the numerator of (4.1) may be brought into the form

$$(B - A^2 - 1)(1 + \Phi^2) - \Phi^2[A^2\Phi^2 + (A^2 - 1)] \tag{4.5}$$

so that, in the case $A^2 > 1$, the first integral (4.1) is valid for

$$B \geqslant A^2 + 1. \tag{4.6}$$

The trajectories in the (Φ, Φ')-phase plane are then closed so that Φ and hence the stress distribution T is periodic. The orbits are symmetric with respect to the Φ- and Φ'-axes and intersect these where

$$\Phi^2 = \frac{B - 2A^2 + \sqrt{B^2 - 4A^2}}{2A^2} \tag{4.7}$$

and

$$\Phi'^2 = \frac{B - A^2 - 1}{(A^2 - 1)^2} \tag{4.8}$$

respectively. The phase portrait for $A^2 = 2$ is displayed in figure 1.

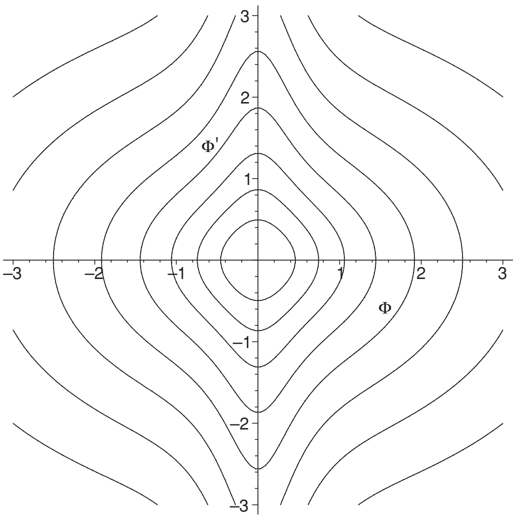

Figure 1. Phase portrait for $A^2 = 2$.

5. Linear approximation

If $T/\sqrt{1+X^2} \ll 1$, then the stress distribution equation (2.22) may be approximated by

$$T_{XX} = \frac{T_{tt}}{(1+X^2)^2} \tag{5.1}$$

and in this case describes uniaxial wave propagation in a linear but inhomogeneous medium. Here, introduction of the base-modulated wave ansatz (3.2) into (5.1) produces, for $A^2 > 1$, a harmonic oscillator equation, namely

$$\Phi'' + \frac{\Phi}{A^2-1} = 0. \tag{5.2}$$

The associated (Φ, Φ')-phase plane is accordingly foliated by closed elliptic trajectories corresponding to its periodic solutions

$$\Phi(s) = \Phi_0 \cos\left(\frac{s-s_0}{\sqrt{A^2-1}}\right). \tag{5.3}$$

In the nonlinear case, the stress–strain law $T = T(X, \epsilon)$ in the implicit form (2.18), for the modulated wave (3.2), produces the expression for the stretch,

$$\epsilon = \frac{1}{2(1+X^2)^{3/2}}\left[\arctan\Phi + \frac{\Phi}{1+\Phi^2}\right]. \tag{5.4}$$

In the present linear approximation with $T/\sqrt{1+X^2} = \Phi \ll 1$, we obtain the uni-directional waves

$$\begin{aligned} T &= \sqrt{1+X^2}\,\Phi(t - A\arctan X) \\ \epsilon &= \frac{1}{(1+X^2)^{3/2}}\Phi(t - A\arctan X), \end{aligned} \tag{5.5}$$

where Φ is given by (5.3). Thus, both the stress and stretch exhibit modulated time-periodic wave behaviour travelling at a local speed which depends on the station X (cf figure 2).

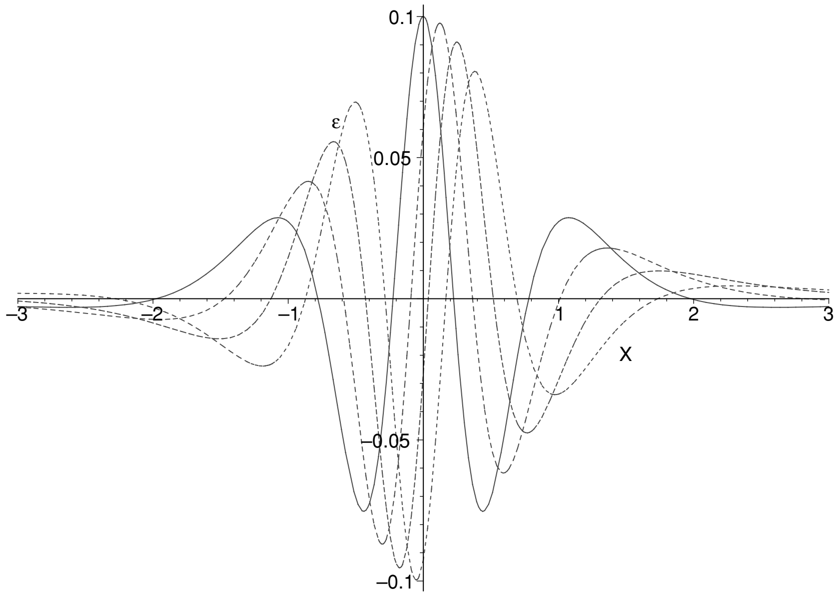

Figure 2. The linearized stretch exhibiting a single modulated wave displayed at various times for $A = 1.01$.

6. Large amplitude limit: phase shift

The nonlinear modulated waves (3.2) exhibit certain intriguing features in a large amplitude limit. These may be detected by direct inspection of the phase diagram associated with Φ. If Φ is 'large', that is,

$$\Phi^2 \gg 1, (4\Phi'^2)^{(1/3)}, \tag{6.1}$$

then (3.5) reduces to

$$\Phi'' + \frac{\Phi}{A^2} = 0. \tag{6.2}$$

It is noted that the frequency of the harmonic oscillator equation (5.2) is larger than that of the harmonic oscillator equation (6.2). In view of this observation, it is natural to consider the initial value problem

$$\Phi(0) = \Phi_0 \geqslant 0, \qquad \Phi'(0) = 0, \tag{6.3}$$

where, *a priori*, Φ is the solution of the nonlinear equation (3.4). A typical numerical solution of this initial value problem for an initial value Φ_0 which compares to unity is shown in figure 3. The phase portrait (cf figure 4) suggests that the oscillator equation (6.2) is actually valid in a significantly larger region than that defined by the restriction (6.1). Indeed, it appears that the solution Φ of (3.4) with trajectory

$$\Phi'^2 = \frac{(1+\Phi^2)^3[-A^2(1+\Phi^2)^2 + B(1+\Phi^2) - 1]}{[A^2(1+\Phi^2)^2 - 1]^2} \tag{6.4}$$

and

$$B = \frac{A^2\left(1+\Phi_0^2\right)^2 + 1}{1+\Phi_0^2} \tag{6.5}$$

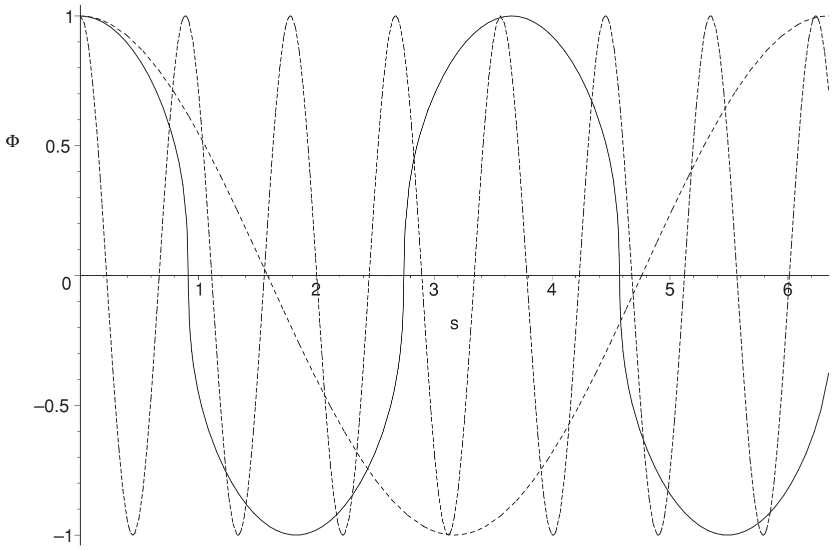

Figure 3. (Numerical) solution of the initial value problem $\Phi_0 = 1$ associated with the nonlinear differential equation (3.4) (solid) and the harmonic oscillator equations (5.2), (6.2) (dashed) for $A^2 = 2$.

tracks the solution Φ_h of the harmonic oscillator equation (6.2) with trajectory

$$\Phi_h'^2 = \frac{\Phi_0^2 - \Phi_h^2}{A^2} \tag{6.6}$$

until Φ reaches a 'critical value' of the order of unity even though condition (6.1) is clearly violated.

In order to make precise the preceding assertion, we now derive an upper bound for the relative 'vertical' deviation of the exact trajectory from the elliptic trajectory, that is,

$$\delta = \left.\frac{\Phi' - \Phi_h'}{\Phi_h'}\right|_{\Phi_h=\Phi}. \tag{6.7}$$

Thus, relations (6.4)–(6.6) imply that

$$\left.\frac{\Phi'^2}{\Phi_h'^2}\right|_{\Phi_h=\Phi} = \left[1 - \frac{1}{A^2(1+\Phi^2)\left(1+\Phi_0^2\right)}\right]\left[1 - \frac{1}{A^2(1+\Phi^2)^2}\right]^{-2}, \tag{6.8}$$

whence

$$1 < \left[1 - \frac{1}{A^2(1+\Phi^2)^2}\right]^{-1} \leqslant (\delta+1)^2 < \left[1 - \frac{1}{A^2(1+\Phi^2)^2}\right]^{-2} \tag{6.9}$$

since $\Phi^2 \leqslant \Phi_0^2$ and $A^2 > 1$. Accordingly, we obtain the upper bound

$$0 < \delta < \frac{1}{(1+\Phi^2)^2 - 1} \tag{6.10}$$

which is independent of both A and Φ_0. The numerical values

$$\delta|_{\Phi=1} < \tfrac{1}{3}, \qquad \delta|_{\Phi=2} < \tfrac{1}{24} \tag{6.11}$$

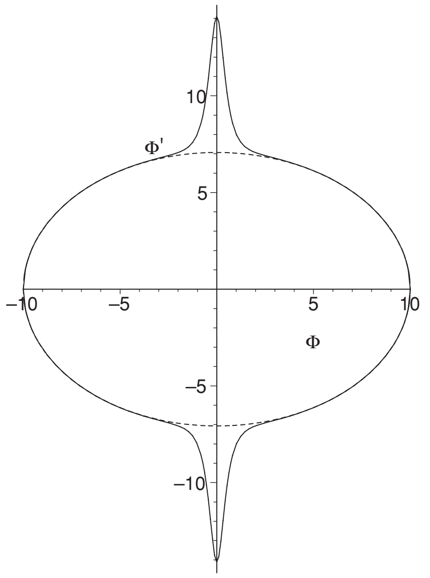

Figure 4. The trajectories of the solution of the initial value problem $\Phi_0 = 10$ associated with the nonlinear differential equation (3.4) (solid) and the harmonic oscillator equation (6.2) (dashed) for $A^2 = 2$.

therefore prove the above conjecture. It is observed that the maximum values of Φ' and Φ'_h are related by

$$\left.\frac{\Phi'}{\Phi'_h}\right|_{\Phi_h=\Phi=0} \approx \frac{A^2}{A^2-1} \tag{6.12}$$

for $\Phi_0^2 \gg 1$. For instance, the latter ratio is 2 if $A^2 = 2$ as illustrated in figure 4.

The preceding analysis implies that Φ and therefore the corresponding modulated wave undergoes a phase shift relative to the (modulated) harmonic wave whenever Φ becomes small. This is indicated in figure 5 for $\Phi_0 = 10$ and $A^2 = 2$. In order to investigate the behaviour of the phase shift for large Φ_0, we focus on the intervals $s \in [0, \sigma]$ and $s \in [0, \sigma_h]$, where σ and σ_h are the quarter-periods of Φ and Φ_h respectively. Since

$$0 < \delta < \frac{1}{A^2(1+\Phi^2)^2 - 1} < \frac{1}{(\sqrt{A^2-1} + |A|\Phi^2)^2}, \tag{6.13}$$

we conclude that

$$\begin{aligned}\Delta\sigma = \sigma_h - \sigma &= \int_{\Phi_0}^{0} \left.\left(\frac{1}{\Phi'_h} - \frac{1}{\Phi'}\right)\right|_{\Phi_h=\Phi} \mathrm{d}\Phi \\ &< \int_{\Phi_0}^{0} \left.\frac{\delta}{\Phi'_h}\right|_{\Phi_h=\Phi} \mathrm{d}\Phi \\ &< \int_{0}^{\Phi_0} \frac{|A|}{\sqrt{\Phi_0^2 - \Phi^2}(\sqrt{A^2-1} + |A|\Phi^2)^2} \mathrm{d}\Phi = K.\end{aligned} \tag{6.14}$$

The latter indefinite integral may be evaluated explicitly and one obtains

$$K \sim \frac{c}{\Phi_0} \qquad \text{as} \quad \Phi_0 \to \infty \tag{6.15}$$

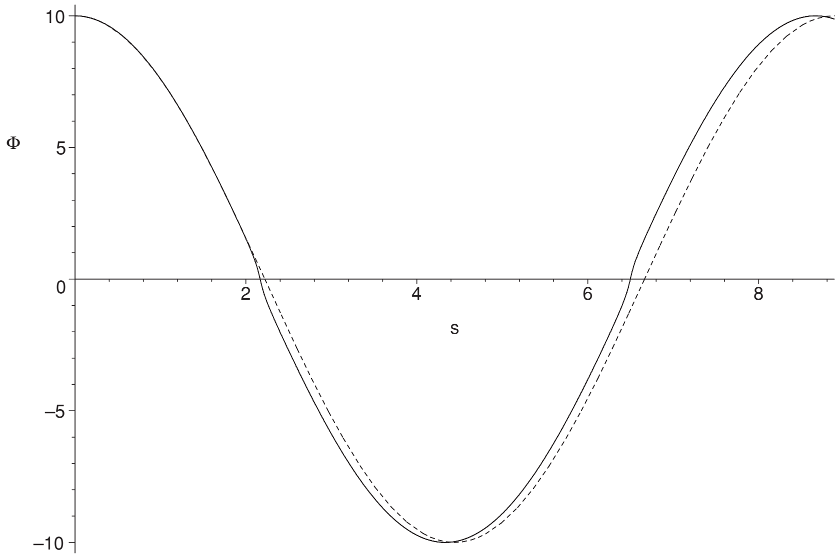

Figure 5. The phase shift of the wave Φ (solid) relative to the harmonic wave Φ_h (dashed) for $\Phi_0 = 10$ and $A^2 = 2$.

for some positive constant c. Accordingly, the phase shift vanishes as $\Phi_0 \to \infty$. Thus, we conclude that Φ approximates a harmonic wave for both 'small' and 'large' initial conditions Φ_0.

7. Superposition of modulated waves

In the linear approximation, a superposition of two identical modulated waves of the type (5.3), (5.5) travelling in opposite directions produces stationary breather solutions given by

$$T = \sqrt{1+X^2}\hat{\Phi}, \qquad \epsilon = \frac{\hat{\Phi}}{(1+X^2)^{3/2}} \tag{7.1}$$

with

$$\begin{aligned}\hat{\Phi} &= \frac{\hat{\Phi}_0}{2}\left[\cos\left(\frac{t - A\arctan X}{\sqrt{A^2-1}}\right) + \cos\left(\frac{t + A\arctan X}{\sqrt{A^2-1}}\right)\right]\\ &= \hat{\Phi}\cos\left(\frac{A\arctan X}{\sqrt{A^2-1}}\right)\cos\left(\frac{t}{\sqrt{A^2-1}}\right).\end{aligned} \tag{7.2}$$

A typical stationary breather solution is displayed in figure 6.

It is seen that this class of breathers constitute solutions with initial data

$$\begin{aligned} T|_{t=0} &= \sqrt{1+X^2}F(X), & T_t|_{t=0} &= 0\\ \epsilon|_{t=0} &= \frac{1}{(1+X^2)^{3/2}}F(X), & \epsilon_t|_{t=0} &= 0,\end{aligned} \tag{7.3}$$

where

$$F(X) = \hat{\Phi}_0\cos\left(\frac{A\arctan X}{\sqrt{A^2-1}}\right). \tag{7.4}$$

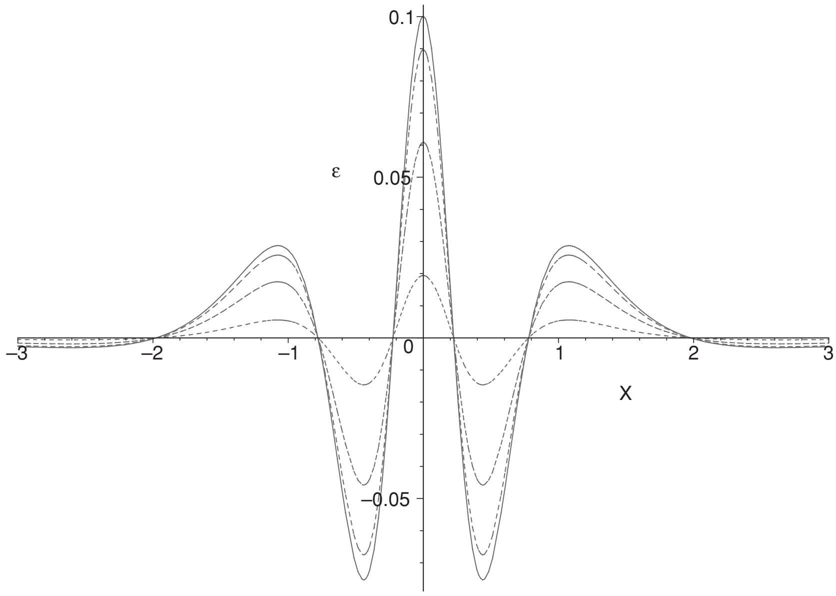

Figure 6. The linearized stretch exhibiting a stationary breather displayed at various times for $A = 1.01$.

In general, a linear superposition of modulated wave forms of the above type may be introduced, namely

$$T = \sqrt{1+X^2}\left[\sum_{n=0}^{\infty}\hat{\Phi}_n \cos\left(\frac{t - A_n \arctan X}{\sqrt{A_n^2 - 1}} + \epsilon_n\right)\right]. \tag{7.5}$$

The nonlinear superposition of modulated waves of the type (3.3), (3.5) by means of the permutability theorem associated with the classical Bäcklund transformation is involved and currently under investigation.

8. Concluding remarks

It is noted that, under the transformation

$$T^* = T\sqrt{1+X^2}, \qquad t^* = t, \qquad X^* = \arctan X, \tag{8.1}$$

the stress propagation equation (2.22) reduces to the nonlinear telegraphy equation

$$T^*_{X^*X^*} - \left(\frac{T^*_{t^*}}{(1+T^{*2})^2}\right)_{t^*} + T^* = 0. \tag{8.2}$$

Accordingly, (8.2) represents yet another integrable avatar of the classical sine-Gordon equation.

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